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Research Article

Eigenvalues And Eigenvectors: Theory and Applications

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Abstract

Eigenvalues and eigenvectors are fundamental concepts in linear algebra that provide important information about the structure and behavior of linear transformations and square matrices. They play a central role in mathematics, physics, engineering, computer science, statistics, and data science. An eigenvector of a matrix is a nonzero vector whose direction remains unchanged when the matrix acts on it, while the corresponding eigenvalue represents the factor by which the vector is scaled. This research article presents a systematic study of the theory of eigenvalues and eigenvectors, including their definitions, characteristic equations, algebraic and geometric multiplicities, eigenspaces, diagonalization, and fundamental properties. The article also examines important applications in systems of linear differential equations, stability analysis, vibration analysis, principal component analysis, Markov chains, quantum mechanics, and graph theory. Particular attention is given to the role of diagonalization and symmetric matrices in simplifying complex mathematical problems. The study demonstrates that eigenvalue-based methods provide a powerful framework for understanding both theoretical and applied problems involving linear transformations.

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1. INTRODUCTION

Linear algebra is one of the most important branches of modern mathematics. It provides the mathematical framework for studying vectors, matrices, linear transformations, systems of equations, and many computational models. Among the most significant concepts in linear algebra are eigenvalues and eigenvectors. These concepts allow a matrix to be studied through its intrinsic directions and scaling behavior rather than merely through its individual entries.

The importance of eigenvalues and eigenvectors extends far beyond theoretical mathematics. They are used to analyze physical systems, solve differential equations, study population models, identify patterns in large datasets, analyze networks, and develop algorithms in computer science. In engineering, eigenvalues are frequently used to study the natural frequencies and stability of mechanical and electrical systems. In statistics and machine learning, eigenvectors and eigenvalues form the mathematical foundation of Principal Component Analysis (PCA), which is used for dimensionality reduction and feature extraction.

The basic idea can be expressed through the equation

$$[A=,]$$

where (A) is an (nn) square matrix, () is a vector, and () is a scalar. The scalar () is called an eigenvalue of (A), and the corresponding nonzero vector () is called an eigenvector.

The study of eigenvalues and eigenvectors provides insight into the internal structure of a matrix. In particular, when a matrix possesses a sufficient number of linearly independent eigenvectors, it can be transformed into diagonal form. This process, known as diagonalization, often converts complicated matrix calculations into simpler scalar operations.

The objective of this article is to present the theoretical

foundations of eigenvalues and eigenvectors and to demonstrate their significance through selected applications.

2. Mathematical Foundations

Let

$$[A=[a_{ij}]]$$

be an (nn) square matrix over a field such as () or ().

A nonzero vector () is called an eigenvector of (A) corresponding to the scalar () if

$$[A=,]$$

Rearranging gives

$$[(A-I)=0,]$$

where (I) denotes the (nn) identity matrix.

For this homogeneous system to possess a nontrivial solution, the matrix (A-I) must be singular. Therefore,

$$[(A-I)=0,]$$

This equation is called the **characteristic equation** of the matrix.

The polynomial

$$[p()=(A-I)]$$

is called the **characteristic polynomial**. Its roots are the eigenvalues of (A).

3. Finding Eigenvalues and Eigenvectors

The calculation of eigenvalues and eigenvectors generally involves the following steps:

1. Form the matrix (A-I).
2. Compute the determinant ((A-I)).
3. Set the determinant equal to zero.
4. Solve the resulting characteristic equation for ().
5. For each eigenvalue, solve [(A-I)=0] to obtain the corresponding eigenvectors.

EXAMPLE: FINDING EIGENVALUES AND EIGENVECTORS

GIVEN MATRIX: Consider the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

CHARACTERISTIC EQUATION: The characteristic equation is $|A - \lambda I| = 0$, $\begin{vmatrix} (4-\lambda) & 1 \\ 2 & (3-\lambda) \end{vmatrix} = 0$

EXPANDING THE DETERMINANT: Expanding: $(4-\lambda)(3-\lambda) - (1)(2) = 0$.

POLYNOMIAL: $\lambda^2 - 7\lambda + 12 - 2 = 0 \rightarrow \lambda^2 - 7\lambda + 10 = 0$.

FACTORING: Factor: $(\lambda - 5)(\lambda - 2) = 0$.

EIGENVALUES: The eigenvalues are $\lambda_1 = 5, \lambda_2 = 2$.

EIGENVECTOR FOR $\lambda_1 = 5$:
For $\lambda_1 = 5$, solve $(A - 5I)\mathbf{v} = 0$. $\rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -v_1 + v_2 = 0 \rightarrow \text{Eigenvector } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 

EIGENVECTOR FOR $\lambda_2 = 2$:
For $\lambda_2 = 2$, solve $(A - 2I)\mathbf{v} = 0$. $\rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 2v_1 + v_2 = 0 \rightarrow \text{Eigenvector } \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ 

EDUCATIONAL REFERENCE MATERIAL - LINEAR ALGEBRA

4. Eigenspaces

For an eigenvalue λ , the eigenspace is defined as

$$[E_{\lambda} = (A - \lambda I).]$$

It consists of the zero vector together with all eigenvectors associated with λ .

The dimension of the eigenspace is called the **geometric multiplicity** of the eigenvalue.

The eigenspace is important because diagonalization depends not merely on the number of eigenvalues but on the number of linearly independent eigenvectors available.

5. Algebraic and Geometric Multiplicity

If λ occurs as a repeated root of the characteristic polynomial, its **algebraic multiplicity** is the number of times it appears as a root.

The **geometric multiplicity** of λ is

$$[(A - \lambda I).]$$

For every eigenvalue,

$$[1.]$$

A matrix is diagonalizable if and only if the total number of linearly independent eigenvectors is equal to the dimension of the matrix.

For an $(n \times n)$ matrix, this means that there must be (n) linearly independent eigenvectors.

A matrix having (n) distinct eigenvalues is automatically diagonalizable because eigenvectors corresponding to distinct eigenvalues are linearly independent. However, repeated eigenvalues require further investigation of their geometric multiplicities.

6. Diagonalization

A matrix (A) is diagonalizable if there exist an invertible matrix (P) and a diagonal matrix (D) such that

$$[A = PDP^{-1}.]$$

The columns of (P) are linearly independent eigenvectors of (A) , while the diagonal entries of (D) are the corresponding eigenvalues.

Diagonalization is useful because powers of (A) can then be calculated efficiently:

$$[A^k = P D^k P^{-1}.]$$

Since (D) is diagonal,

$$[D^k = .]$$

Consequently, a difficult matrix-power calculation can be reduced to the computation of powers of individual eigenvalues.

7. Important Properties of Eigenvalues

Eigenvalues satisfy several important relationships with the matrix.

FUNDAMENTAL EIGENVALUE PROPERTIES

7.1 Trace

The sum of the eigenvalues, counted with algebraic multiplicity, is equal to the trace:

$$\text{tr}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n.$$

$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ $\lambda_1 = 2, \text{tr}(A) = 3 + 4 = 7$
 $\lambda_2 = 5, \lambda_1 + \lambda_2 = 2 + 5 = 7$

7.2 Determinant

The determinant is equal to the product of the eigenvalues: $\det(A) = \lambda_1 \lambda_2 \dots \lambda_n.$

$$\det(A) = (3)(4) - (1)(2) = 10$$

$$\lambda_1 \lambda_2 = (2)(5) = 10$$

Thus, a matrix is singular if and only if zero is an eigenvalue.

$B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\det(B) = 0$
Eigenvalues $\{3, 0\}$

7.3 Triangular Matrices

For a triangular matrix, the eigenvalues are its diagonal entries.

$U = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ $L = \begin{bmatrix} 2 & 0 \\ 5 & -1 \end{bmatrix}$

Eigenvalues $\lambda_1 = 3, \lambda_2 = 4$ Eigenvalues $\lambda_1 = 2, \lambda_2 = -1$

7.4 Inverse Matrix

If A is invertible and λ is an eigenvalue of A , then λ^{-1} is an eigenvalue of A^{-1} .

For $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ with $\lambda_1 = 2, \lambda_2 = 5$. $A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$.

Eigenvalues of A^{-1} are $\lambda_1^{-1} = 1/2, \lambda_2^{-1} = 1/5$.

EDUCATIONAL REFERENCE MATERIAL - LINEAR ALGEBRA PROPERTIES

8. Symmetric Matrices and Orthogonal Diagonalization

Real symmetric matrices have particularly important properties. If

$$[A = A^T,]$$

then all eigenvalues of (A) are real, and eigenvectors corresponding to distinct eigenvalues are orthogonal. The **Spectral Theorem** states that every real symmetric matrix can be orthogonally diagonalized. Thus, there exists an orthogonal matrix (Q) such that $[A = QDQ^T.]$ This property has significant applications in quadratic forms, optimization,

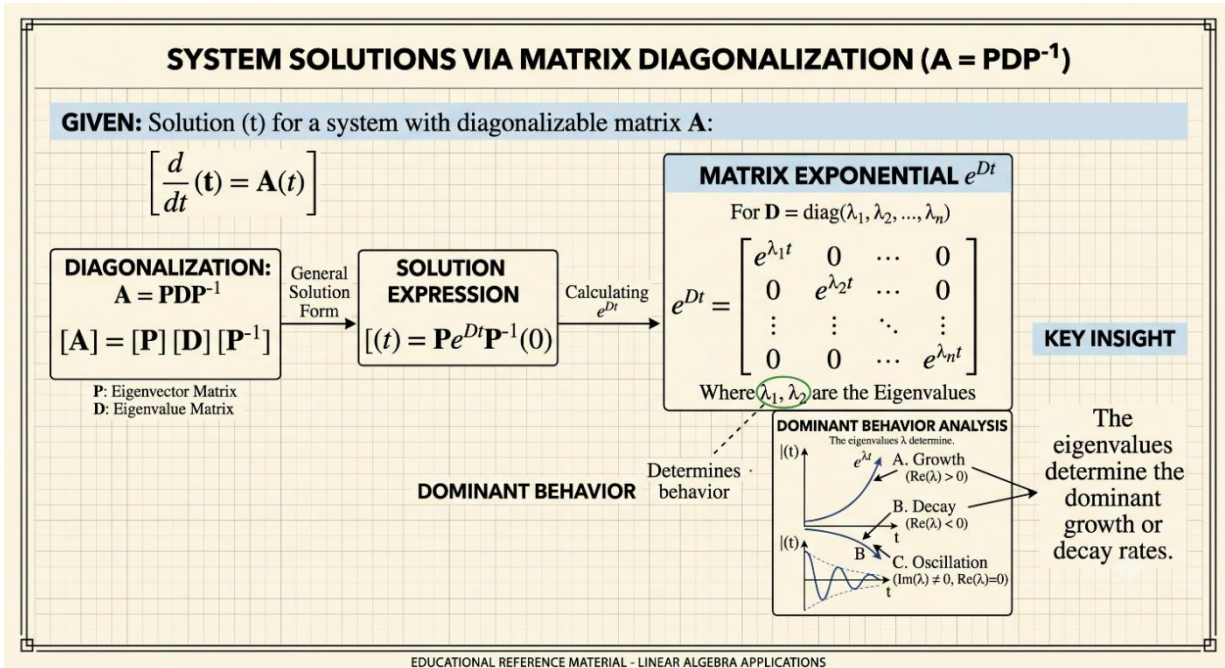
statistics, numerical analysis, and physics. The spectral decomposition can be written as

$$[A = QDQ^T.]$$

Because $(Q^{-1}) = Q^T$, calculations involving symmetric matrices are often considerably simpler and numerically more stable.

9. Application to Systems of Differential Equations

Eigenvalues and eigenvectors are important tools for solving systems of linear differential equations.



If all eigenvalues have negative real parts, solutions generally decay toward the equilibrium. If at least one eigenvalue has positive real part, the system may exhibit unstable behavior.

10. Application in Stability Analysis

Eigenvalues provide a fundamental criterion for studying the stability of equilibrium points in dynamical systems.

For a linear system

$$[\dot{x}] = Ax,$$

the real parts of the eigenvalues determine the behavior near the equilibrium.

- If all eigenvalues have negative real parts, the equilibrium is asymptotically stable.
- If at least one eigenvalue has a positive real part, the equilibrium is unstable.
- Eigenvalues with zero real parts require additional analysis.

This principle is widely used in control theory, engineering systems, economics, population models, and mathematical biology.

11. Application in Vibration Analysis

Eigenvalue methods are extensively used in mechanical and structural engineering.

A simplified vibration model can be written as

$$[M]\ddot{x} + [K]x = 0,$$

where (M) is the mass matrix and (K) is the stiffness matrix.

Assuming a solution of the form

$$x(t) = e^{i\omega t},$$

leads to the generalized eigenvalue problem

$$[K - \omega^2 M]x = 0.$$

The eigenvalues are related to the squares of the natural frequencies, while the eigenvectors describe the corresponding mode shapes.

This technique is essential in the design and analysis of buildings, bridges, vehicles, aircraft, machines, and other structures.

12. Application in Principal Component Analysis

Eigenvalues and eigenvectors are central to Principal

Component Analysis (PCA).

Suppose a dataset has covariance matrix (C). The eigenvectors of (C) define the principal directions of the data, while the corresponding eigenvalues indicate the amount of variance represented in those directions.

If $[v_1, v_2, \dots, v_n]$ then the eigenvector associated with (λ_1) represents the direction of maximum variance.

The first few eigenvectors can therefore be selected to represent the dataset in a lower-dimensional space while retaining a large proportion of its information.

PCA is widely applied in data visualization, image processing, pattern recognition, signal processing, and machine learning.

13. Application in Markov Chains

Eigenvalues also play an important role in the study of Markov chains.

A transition matrix (P) describes the probabilities of moving between different states. Long-term behavior can often be analyzed using the eigenvalue (1) and its corresponding eigenvectors.

For a suitable stochastic matrix, a stationary distribution (π) satisfies $[\pi P = \pi]$

Thus, (π) is an eigenvector associated with the eigenvalue (1).

The remaining eigenvalues help determine the rate at which the Markov chain approaches its stationary behavior. This has applications in probability, economics, population studies, queueing systems, and ranking algorithms.

14. Application in Graph Theory

Eigenvalues of matrices associated with graphs provide valuable information about network structure.

For a graph (G), important matrices include the adjacency matrix (A) and graph Laplacian (L).

The eigenvalues of the adjacency matrix can provide information about connectivity and structural characteristics of the graph. The Laplacian matrix is particularly important in spectral graph theory.

The number of zero eigenvalues of the graph Laplacian equals the number of connected components of an undirected graph. The second-smallest Laplacian eigenvalue, known as the algebraic connectivity, provides information about how strongly connected the graph is.

These ideas have applications in network analysis, clustering, communication networks, social networks, and computer science.

15. Application in Quantum Mechanics

Eigenvalues and eigenvectors have a fundamental interpretation in quantum mechanics.

Physical quantities such as energy are represented mathematically by operators. Measurement outcomes correspond to eigenvalues of the associated operators, while the corresponding eigenvectors represent possible states associated with those outcomes.

For an operator (A), $[A=]$

where $()$ represents a state and $()$ represents a possible measured value.

The eigenvalue framework therefore provides the mathematical basis for describing observable quantities and quantum states.

16. Eigenvalues in Numerical Methods

Eigenvalue problems are also important in numerical analysis. For large matrices, exact symbolic calculation of characteristic polynomials is often impractical. Numerical methods are therefore used to approximate eigenvalues and eigenvectors.

Common methods include:

- Power Method
- Inverse Power Method
- Rayleigh Quotient Iteration
- Jacobi Method
- QR Algorithm
- Lanczos Method

The Power Method is particularly useful for approximating the dominant eigenvalue. Starting with a suitable nonzero vector $(_0)$, repeated multiplication gives

$$[_{k+1}] = A[_k]$$

Under appropriate conditions, the normalized vectors converge toward an eigenvector associated with the eigenvalue of largest magnitude.

Modern computational software uses advanced eigenvalue algorithms to handle large-scale scientific and engineering problems.

17. DISCUSSION

The theory of eigenvalues and eigenvectors demonstrates a deep connection between algebraic structure and practical behavior. The eigenvalues summarize important characteristics of a transformation, while eigenvectors identify the directions in which the transformation acts by simple scaling.

Diagonalization provides one of the most powerful consequences of this theory. When a matrix has a complete set of linearly independent eigenvectors, complicated operations involving the matrix can be simplified substantially.

However, not every matrix is diagonalizable. Repeated eigenvalues may not provide enough linearly independent eigenvectors. In such cases, other approaches, including Jordan canonical form, may be required.

Symmetric matrices represent an especially useful class because they always possess real eigenvalues and an orthogonal basis of eigenvectors. Their favorable structure explains their extensive use in optimization, statistics, numerical computation, and physical modeling.

The applications discussed in this article demonstrate that eigenvalue methods are not limited to abstract mathematical theory. They provide practical tools for solving differential equations, understanding stability, analyzing vibrations, reducing data dimensions, studying networks, and interpreting physical systems.

18. CONCLUSION

Eigenvalues and eigenvectors constitute one of the central pillars of linear algebra. The equation $[A=]$ provides a simple but powerful framework for identifying special directions and scaling factors associated with a linear transformation.

The characteristic equation determines the eigenvalues, while the corresponding null spaces determine the eigenvectors. Concepts such as algebraic multiplicity, geometric multiplicity, eigenspaces, and diagonalization provide a deeper understanding of matrix structure.

The importance of eigenvalues and eigenvectors is evident across numerous disciplines. They are used in differential equations to determine system behavior, in engineering to identify natural frequencies, in statistics to determine principal components, in Markov chains to analyze long-term behavior, in graph theory to study networks, and in quantum mechanics to describe measurable physical quantities.

Therefore, the study of eigenvalues and eigenvectors represents more than a fundamental topic in linear algebra. It is a versatile mathematical framework that connects theoretical mathematics with practical applications across science, engineering, technology, and data analysis.

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Manisha Sahu completed her postgraduate degree in Mathematics from Guru Ghasidas University (GGU). She qualified the CGSET examination on her first attempt. She has served as a Mathematics lecturer at Government Naveen College, Ghotiya, Balod, and is currently working as a Guest Lecturer in the Department of Mathematics at Government Naveen College, Dadhi, Bemetara. She has a deep interest in Mathematics, especially in mathematical theorems and their applications. She enjoys teaching Mathematics and helping students understand mathematical concepts in a simple and clear way. Her passion for the subject inspires her to continue learning and sharing her knowledge with others.